NONCLASSICAL REPRESENTATIONS OF THE NONSTANDARD DEFORMATIONS $U'_q(\mathrm{so}_n)$, $U_q(\mathrm{iso}_n)$ AND $U'_q(\mathrm{so}_{n,1})$

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Abstract

The aim of this paper is to announce the results on irreducible nonclassical type representations of the nonstandard q-deformations $U'_q(\mathrm{so}_n)$, $U_q(\mathrm{iso}_n)$ and $U'_q(\mathrm{so}_{n,1})$ of the universal enveloping algebras of the Lie algebras $\mathrm{so}(n,\mathbf{C})$, iso_n and $\mathrm{so}_{n,1}$ when q is a real number (the algebra $U'_q(\mathrm{so}_{n,1})$ is a real form of the algebra $U'_q(\mathrm{so}_{n+1})$). These representations are characterized by the properties that they are singular at the point $q=1$.

1. Introduction

Quantum orthogonal groups, quantum Lorentz group and their corresponding quantum algebras are of special interest for modern physics [1-3]. M. Jimbo [4] and V. Drinfeld [5] defined q-deformations (quantum algebras) $U_q(g)$ for all simple complex Lie algebras g by means of Cartan subalgebras and root subspaces (see also [6]). Reshetikhin, Takhtajan and Faddeev [7] defined quantum algebras $U_q(g)$ in terms of the universal R-matrix. However, these approaches do not give a satisfactory presentation of the quantum algebra $U_q(\mathrm{so}(n,\mathbf{C}))$ from a viewpoint of some problems in quantum physics and representation theory. When considering representations of the quantum groups $SO_q(n+1)$ and $SO_q(n,1)$ we are interested in reducing them onto the quantum subgroup $SO_q(n)$. This reduction would give the analogue of the Gel'fand-Tsetlin basis for these representations. However, definitions of quantum algebras mentioned above do not allow the inclusions $U_q(\mathrm{so}(n+1,\mathbf{C}))\supset U_q(\mathrm{so}(n,\mathbf{C}))$ and $U_q(\mathrm{so}_{n,1})\supset U_q(\mathrm{so}_n)$. To be able to exploit such reductions we have to consider q-deformations of the Lie algebra $\mathrm{so}(n+1,\mathbf{C})$ defined in terms of the generators $I_{k,k-1}=E_{k,k-1}-E_{k-1,k}$ (where E_{is} is the matrix with elements $(E_{is})_{rt}=\delta_{ir}\delta_{st}$) rather than by means of Cartan subalgebras and root elements. To construct such deformations we have to deform trilinear relations for elements $I_{k,k-1}$ instead of Serre's relations (as in the case of Jimbo's quantum algebras). As a result, we obtain the associative algebra which will be denoted as $U'_q(\mathrm{so}(n,\mathbf{C}))$.

These q-deformations were first constructed in [8]. They permit one to construct the reductions of $U'_q(\mathrm{so}_{n,1})$ and $U'_q(\mathrm{so}_{n+1})$ onto $U'_q(\mathrm{so}_n)$. The q-deformed algebra $U'_q(\mathrm{so}(n,\mathbf{C}))$ leads for $n=3$ to the q-deformed algebra $U'_q(\mathrm{so}(3,\mathbf{C}))$ defined by D. Fairlie [9]. The cyclically symmetric algebra, similar to Fairlie's one, was also considered somewhat earlier by Odesskii [10]. The algebra $U'_q(\mathrm{so}(3,\mathbf{C}))$ allows us to construct the noncompact quantum algebra $U'_q(\mathrm{so}_{2,1})$. The algebra $U'_q(\mathrm{so}(4,\mathbf{C}))$ is a q-deformation of the Lie algebra $\mathrm{so}(4,\mathbf{C})$ given by means of usual bilinear commutation relations between the elements I_{ji}, $1\le i<j\le 4$. In the case of the classical Lie algebra $\mathrm{so}(4,\mathbf{C})$ one has $\mathrm{so}(4,\mathbf{C})=\mathrm{so}(3,\mathbf{C})+\mathrm{so}(3,\mathbf{C})$, while in the case of our q-deformation $U'_q(\mathrm{so}(4,\mathbf{C}))$ this is not the case.

In the classical case, the imbedding $SO(n) \subset SU(n)$ (and its infinitesimal analogue) is of great importance for nuclear physics and in the theory of Riemannian spaces. It is well known that in the framework of Drinfeld–Jimbo quantum groups and algebras one cannot construct the corresponding imbedding. The algebra $U'_q(\mathrm{so}(n,\mathbf{C}))$ allows to define such an imbedding [11], that is, it is possible to define the imbedding $U'_q(\mathrm{so}(n,\mathbf{C})) \subset U_q(\mathrm{sl}_n)$, where $U_q(\mathrm{sl}_n)$ is the Drinfeld-Jimbo quantum algebra.

As a disadvantage of the algebra $U'_q(\mathrm{so}(n,\mathbf{C}))$ we have to mention the difficulties with Hopf algebra structure. Nevertheless, $U'_q(\mathrm{so}(n,\mathbf{C}))$ turns out to be a coideal in $U_q(\mathrm{sl}_n)$.

Finite dimensional irreducible representations of the algebra $U'_q(\mathrm{so}(n,\mathbf{C}))$ were constructed in [8]. The formulas of action of the generators of $U'_q(\mathrm{so}(n,\mathbf{C}))$ upon the basis (which is a q-analogue of the Gel'fand–Tsetlin basis) are given there. A proof of these formulas and some their corrections were given in [12]. However, finite dimensional irreducible representations described in [8] and [12] are representations of the classical type. They are q-deformations of the corresponding irreducible representations of the Lie algebra $\mathrm{so}(n,\mathbf{C})$, that is, at $q \to 1$ they turn into representations of $\mathrm{so}(n,\mathbf{C})$.

The algebra $U'_q(\mathrm{so}(n,\mathbf{C}))$ has other classes of finite dimensional irreducible representations which have no classical analogue. These representations are singular at the limit $q \to 1$. One of the aims of this paper is to describe these representations of $U'_q(\mathrm{so}(n,\mathbf{C}))$. Note that the description of these representations for the algebra $U'_q(\mathrm{so}(3,\mathbf{C}))$ is given in [13]. A classification of irreducible $*$-representations of real forms of the algebra $U'_q(\mathrm{so}(3,\mathbf{C}))$ is given in [14].

There exists an algebra, closely related to the algebra $U'_q(\mathrm{so}(n,\mathbf{C}))$, which is a q-deformation of the universal enveloping algebra $U(\mathrm{iso}_n)$ of the Lie algebra iso_n of the Euclidean group $ISO(n)$ (see [15]). It is denoted as $U_q(\mathrm{iso}_n)$. Irreducible representations of the classical type of the algebra $U_q(\mathrm{iso}_n)$ were described in [15]. A proof of the corresponding formulas was given in [16]. However, the algebra $U_q(\mathrm{iso}_n)$, $q \in \mathbf{R}$, has irreducible representations of the nonclassical type. A description of these representations is the second aim of this paper. Note that the description of these representations for $U_q(\mathrm{iso}_2)$ is given in [17]. A classification of irreducible $*$-representations of $U_q(\mathrm{iso}_2)$ is obtained in [18]. The last aim of this paper is to describe irreducible representations of nonclassical type of the algebra $U'_q(\mathrm{so}_{n,1})$ which is a real form of the algebra $U'_q(\mathrm{so}(n+1,\mathbf{C}))$. Representations of the classical type of this algebra are described in [8] and [20].

We assume throughout the paper that q is a fixed positive number. Thus, we give formulas for representations for these values of q. However, these representations can be considered for any values of q not coinciding with a root of unity. For this we have to treat appropriately square roots in formulas for representations or to rescale basis vector in such a way that formulas for representations would not contain square roots.

For convenience, we denote the Lie algebra $\mathrm{so}(n,\mathbf{C})$ by so_n and the algebra $U'_q(\mathrm{so}(n,\mathbf{C}))$ by $U'_q(\mathrm{so}_n)$.

2. The q-deformed algebras $U'_q(\mathrm{so}_n)$ and $U_q(\mathrm{iso}_n)$

In our approach [8] to the q-deformation of the algebras $U(\mathrm{so}_n)$ we define the q-deformed algebras $U'_q(\mathrm{so}(n,\mathbf{C}))$ as the associate algebra (with a unit) generated by the elements $I_{i,i-1}$, $i = 2,3,...,n$ satisfying the defining relations

$$I_{i,i-1}I^2_{i-1,i-2} - (q+q^{-1})I_{i-1,i-2}I_{i,i-1}I_{i-1,i-2} + I^2_{i-1,i-2}I_{i,i-1} = -I_{i,i-1}, \qquad (1)$$

$$I^2_{i,i-1}I_{i-1,i-2} - (q+q^{-1})I_{i,i-1}I_{i-1,i-2}I_{i,i-1} + I_{i-1,i-2}I^2_{i,i-1} = -I_{i-1,i-2}, \qquad (2)$$

$$[I_{i,i-1}, I_{j,j-1}] = 0, \quad |i-j| > 1, \tag{3}$$

where [.,.] denotes usual commutator. Obviously, in the limit $q \to 1$ formulas (1)–(3) give the relations

$$I_{i,i-1}I^2_{i-1,i-2} - 2I_{i-1,i-2}I_{i,i-1}I_{i-1,i-2} + I^2_{i-1,i-2}I_{i,i-1} = -I_{i,i-1},$$

$$I^2_{i,i-1}I_{i-1,i-2} - 2I_{i,i-1}I_{i-1,i-2}I_{i,i-1} + I_{i-1,i-2}I^2_{i,i-1} = -I_{i-1,i-2},$$

$$[I_{i,i-1}, I_{j,j-1}] = 0, \quad |i-j| > 1,$$

defining the universal enveloping algebra $U(\mathrm{so}_n)$. Note also that relations (1) and (2) principally differ from the q-deformed Serre relations in the approach of Jimbo [4] and Drinfeld [5] to quantum orthogonal algebras by a presence of nonzero right hand side and by possibility of the reduction

$$U'_q(\mathrm{so}_n) \supset U'_q(\mathrm{so}_{n-1}) \supset \cdots \supset U'_q(\mathrm{so}_3).$$

Recall that in the standard Jimbo–Drinfeld approach to the definition of quantum algebras, the algebras $U_q(\mathrm{so}_{2m})$ and the algebras $U_q(\mathrm{so}_{2m+1})$ are distinct series of quantum algebras which are constructed independently of each other.

Various real forms of the algebras $U'_q(\mathrm{so}_n)$ are obtained by imposing corresponding $*$-structures (antilinear antiautomorphisms). The compact real form $U'_q(\mathrm{so}(n))$ is defined by the $*$-structure

$$I^*_{i,i-1} = -I_{i,i-1}, \quad i = 2, 3, ..., n.$$

The noncompact q-deformed algebras $U'_q(\mathrm{so}_{p,r})$ where $r = n-p$ are singled out respectively by means of the $*$-structures

$$I^*_{i,i-1} = -I_{i,i-1}, \quad i \neq p+1, \quad i \leq n, \quad I^*_{p+1,p} = I_{p+1,p}.$$

Among the noncompact real q-algebras $U'_q(\mathrm{so}_{p,r})$, the algebras $U'_q(\mathrm{so}_{n-1,1})$ (a q-analogue of the Lorentz algebras) are of special interest.

We also define the algebra $U_q(\mathrm{iso}_n)$ which is a nonstandard deformation of the universal enveloping algebra of the Lie algebra iso_n of the Euclidean Lie group $ISO(n)$. It is the associative algebra (with a unit) generated by the elements $I_{21}, I_{32}, \cdots, I_{n,n-1}, T_n$ such that the elements $I_{21}, I_{32}, \cdots, I_{n,n-1}$ satisfy the defining relations of the subalgebra $U'_q(\mathrm{so}_n)$ and the additional defining relations are

$$I^2_{n,n-1}T_n - (q+q^{-1})I_{n,n-1}T_nI_{n,n-1} + T_nI^2_{n,n-1} = -T_n,$$

$$T^2_nI_{n,n-1} - (q+q^{-1})T_nI_{n,n-1}T_n + I_{n,n-1}T^2_n = 0,$$

$$[I_{k,k-1}, T_n] \equiv I_{k,k-1}T_n - T_nI_{k,k-1} = 0 \quad \text{if} \quad k < n$$

(see [15]). If $q = 1$, then these relations define the classical algebra $U(\mathrm{iso}_n)$.

Let us note that the defining relations for $U_q(\mathrm{iso}_n)$ can be expressed by bilinear relations [16]. As an example, we consider the algebra $U_q(\mathrm{iso}_2)$ (see [16] and [18]). This algebra is generated by the elements I_{21} and T_2. Setting $T_1 = [I_{21}, T_2]_q \equiv q^{1/2}I_{21}T_2 - q^{-1/2}T_2I_{21}$, we obtain from the two defining relations the bilinear relations

$$[I_{21}, T_2]_q = T_1, \quad [T_1, I_{21}]_q = T_2, \quad [T_2, T_1]_q = 0$$

which are also defining relations for $U_q(\mathrm{iso}_2)$. Note that the elements T_1 and T_2 corresponding to infinitesimal generators of shifts in the Lie algebra iso_2 do not commute (they q-commute, that is, $q^{1/2}T_2T_1 - q^{-1/2}T_1T_2 = 0$). A similar picture we have for the algebra $U_q(\mathrm{iso}_n)$.

3. Finite dimensional classical type representations of $U'_q(\mathrm{so}_n)$

In this section we describe (in the framework of a q-analogue of Gel'fand–Tsetlin formalism) irreducible finite dimensional representations of the algebras $U'_q(\mathrm{so}_n)$, $n \geq 3$, which are q-deformations of the finite dimensional irreducible representations of the Lie algebra so_n. They are given by sets $\mathbf{m}_n$ consisting of $\{n/2\}$ numbers $m_{1,n}, m_{2,n}, ..., m_{\{n/2\},n}$ (here $\{n/2\}$ denotes integral part of $n/2$) which are all integral or all half-integral and satisfy the dominance conditions

$$m_{1,2p+1} \geq m_{2,2p+1} \geq ... \geq m_{p,2p+1} \geq 0, \tag{4}$$

$$m_{1,2p} \geq m_{2,2p} \geq ... \geq m_{p-1,2p} \geq |m_{p,2p}| \tag{5}$$

for $n = 2p+1$ and $n = 2p$, respectively. These representations are denoted by $T_{\mathbf{m}_n}$. For a basis in a representation space we take the q-analogue of Gel'fand–Tsetlin basis which is obtained by successive reduction of the representation $T_{\mathbf{m}_n}$ to the subalgebras $U'_q(\mathrm{so}_{n-1})$, $U'_q(\mathrm{so}_{n-2})$, $\cdots$, $U'_q(\mathrm{so}_3)$, $U'_q(\mathrm{so}_2) := U(\mathrm{so}_2)$. As in the classical case, its elements are labelled by Gel'fand–Tsetlin tableaux

$$\{\xi_n\} \equiv \left\{ \begin{matrix} \mathbf{m}_n \\ \mathbf{m}_{n-1} \\ \dots \\ \mathbf{m}_2 \end{matrix} \right\} \equiv \{\mathbf{m}_n, \xi_{n-1}\} \equiv \{\mathbf{m}_n, \mathbf{m}_{n-1}, \xi_{n-2}\}, \tag{6}$$

where the components of $\mathbf{m}_n$ and $\mathbf{m}_{n-1}$ satisfy the "betweenness" conditions

$$m_{1,2p+1} \geq m_{1,2p} \geq m_{2,2p+1} \geq m_{2,2p} \geq ... \geq m_{p,2p+1} \geq m_{p,2p} \geq -m_{p,2p+1}, \tag{7}$$

$$m_{1,2p} \geq m_{1,2p-1} \geq m_{2,2p} \geq m_{2,2p-1} \geq ... \geq m_{p-1,2p-1} \geq |m_{p,2p}|. \tag{8}$$

The basis element defined by tableau $\{\xi_n\}$ is denoted as $|\{\xi_n\}\rangle$ or simply as $|\xi_n\rangle$.

It is convenient to introduce the so-called l-coordinates

$$l_{j,2p+1} = m_{j,2p+1} + p - j + 1, \qquad l_{j,2p} = m_{j,2p} + p - j, \tag{9}$$

for the numbers $m_{i,k}$. In particular, $l_{1,3} = m_{1,3} + 1$ and $l_{1,2} = m_{1,2}$. The operator $T_{\mathbf{m}_n}(I_{2p+1,2p})$ of the representation $T_{\mathbf{m}_n}$ of $U'_q(\mathrm{so}_n)$ acts upon Gel'fand–Tsetlin basis elements, labelled by (6), by the formula

$$T_{\mathbf{m}_n}(I_{2p+1,2p})|\xi_n\rangle = \sum_{j=1}^{p} \frac{A^j_{2p}(\xi_n)}{q^{l_{j,2p}} + q^{-l_{j,2p}}} |(\xi_n)^{+j}_{2p}\rangle - \sum_{j=1}^{p} \frac{A^j_{2p}((\xi_n)^{-j}_{2p})}{q^{l_{j,2p}} + q^{-l_{j,2p}}} |(\xi_n)^{-j}_{2p}\rangle \tag{10}$$

and the operator $T_{\mathbf{m}_n}(I_{2p,2p-1})$ of the representation $T_{\mathbf{m}_n}$ acts as

$$T_{\mathbf{m}_n}(I_{2p,2p-1})|\xi_n\rangle = \sum_{j=1}^{p-1} \frac{B^j_{2p-1}(\xi_n)}{[2l_{j,2p-1} - 1][l_{j,2p-1}]} |(\xi_n)^{+j}_{2p-1}\rangle$$

$$- \sum_{j=1}^{p-1} \frac{B^j_{2p-1}((\xi_n)^{-j}_{2p-1})}{[2l_{j,2p-1} - 1][l_{j,2p-1} - 1]} |(\xi_n)^{-j}_{2p-1}\rangle + \mathrm{i}\, C_{2p-1}(\xi_n)|\xi_n\rangle. \tag{11}$$

In these formulas, $(\xi_n)_k^{\pm j}$ means the tableau (6) in which j-th component $m_{j,k}$ in $\mathbf{m}_k$ is replaced by $m_{j,k} \pm 1$. The coefficients A_{2p}^j, B_{2p-1}^j, C_{2p-1} in (10) and (11) are given by the expressions

$$A_{2p}^j(\xi_n) = \left(\frac{\prod_{i=1}^{p} [l_{i,2p+1} + l_{j,2p}][l_{i,2p+1} - l_{j,2p} - 1] \prod_{i=1}^{p-1} [l_{i,2p-1} + l_{j,2p}][l_{i,2p-1} - l_{j,2p} - 1]}{\prod_{i \neq j}^{p} [l_{i,2p} + l_{j,2p}][l_{i,2p} - l_{j,2p}][l_{i,2p} + l_{j,2p} + 1][l_{i,2p} - l_{j,2p} - 1]} \right)^{1/2}, \tag{12}$$

and

$$B_{2p-1}^j(\xi_n) = \left(\frac{\prod_{i=1}^{p} [l_{i,2p} + l_{j,2p-1}][l_{i,2p} - l_{j,2p-1}] \prod_{i=1}^{p-1} [l_{i,2p-2} + l_{j,2p-1}][l_{i,2p-2} - l_{j,2p-1}]}{\prod_{i \neq j}^{p-1} [l_{i,2p-1} + l_{j,2p-1}][l_{i,2p-1} - l_{j,2p-1}][l_{i,2p-1} + l_{j,2p-1} - 1][l_{i,2p-1} - l_{j,2p-1} - 1]} \right)^{1/2}, \tag{13}$$

$$C_{2p-1}(\xi_n) = \frac{\prod_{s=1}^{p} [l_{s,2p}] \prod_{s=1}^{p-1} [l_{s,2p-2}]}{\prod_{s=1}^{p-1} [l_{s,2p-1}][l_{s,2p-1} - 1]}, \tag{14}$$

where numbers in square brackets mean q-numbers defined by

$$[a] := \frac{q^a - q^{-a}}{q - q^{-1}}.$$

In particular,

$$T_{\mathbf{m}_n}(I_{3,2})|\xi_n\rangle = \frac{1}{q^{m_{1,2}} + q^{-m_{1,2}}}(([l_{1,3} + m_{1,2}][l_{1,3} - m_{1,2} - 1])^{1/2} |(\xi_n)_2^{+1}\rangle -$$

$$-([l_{1,3} + m_{1,2} - 1][l_{1,3} - m_{1,2}])^{1/2} |(\xi_n)_2^{-1}\rangle),$$

$$T_{\mathbf{m}_n}(I_{2,1})|\xi_n\rangle = \mathrm{i}[m_{1,2}]|\xi_n\rangle,$$

It is seen from (9) that C_{2p-1} in (14) identically vanishes if $m_{p,2p} \equiv l_{p,2p} = 0$.

A proof of the fact that formulas (10)-(14) indeed determine a representation of $U_q'(\mathrm{so}_n)$ is given in [12].

4. Finite dimensional nonclassical type representations of $U_q'(\mathrm{so}_n)$

The representations of the previous section are called representations of the classical type, since under the limit $q \to 1$ the operators $T_{\mathbf{m}_n}(I_{j,j-1})$ turn into the corresponding operators $T_{\mathbf{m}_n}(I_{j,j-1})$ for irreducible finite dimensional representations with highest weights $\mathbf{m}_n$ of the Lie algebra so_n.

The algebra $U_q'(\mathrm{so}_n)$ also has irreducible finite dimensional representations T of nonclassical type, that is, such that the operators $T(I_{j,j-1})$ have no classical limit $q \to 1$. They are given by sets $\epsilon := (\epsilon_2, \epsilon_3, \cdots, \epsilon_n)$, $\epsilon_i = \pm 1$, and by sets $\mathbf{m}_n$ consisting of $\{n/2\}$ **half-integral** numbers $m_{1,n}, m_{2,n}, \cdots, m_{\{n/2\},n}$ (here $\{n/2\}$ denotes integral part of $n/2$) that satisfy the dominance conditions

$$m_{1,2p+1} \geq m_{2,2p+1} \geq ... \geq m_{p,2p+1} \geq 1/2, \tag{15}$$

$$m_{1,2p} \geq m_{2,2p} \geq ... \geq m_{p-1,2p} \geq m_{p,2p} \geq 1/2 \tag{16}$$

for $n = 2p+1$ and $n = 2p$, respectively. These representations are denoted by $T_{\epsilon,\mathbf{m}_n}$.

For a basis in the representation space we use the analogue of the basis of the previous section. Its elements are labelled by tableaux

$$\{\xi_n\} \equiv \left\{ \begin{array}{c} \mathbf{m}_n \\ \mathbf{m}_{n-1} \\ \dots \\ \mathbf{m}_2 \end{array} \right\} \equiv \{\mathbf{m}_n, \xi_{n-1}\} \equiv \{\mathbf{m}_n, \mathbf{m}_{n-1}, \xi_{n-2}\}, \tag{17}$$

where the components of $\mathbf{m}_n$ and $\mathbf{m}_{n-1}$ satisfy the "betweenness" conditions

$$m_{1,2p+1} \ge m_{1,2p} \ge m_{2,2p+1} \ge m_{2,2p} \ge ... \ge m_{p,2p+1} \ge m_{p,2p} \ge 1/2, \tag{18}$$

$$m_{1,2p} \ge m_{1,2p-1} \ge m_{2,2p} \ge m_{2,2p-1} \ge ... \ge m_{p-1,2p-1} \ge m_{p,2p}. \tag{19}$$

The basis element defined by tableau $\{\xi_n\}$ is denoted as $|\xi_n\rangle$.

As in the previous section, it is convenient to introduce the l-coordinates

$$l_{j,2p+1} = m_{j,2p+1} + p - j + 1, \qquad l_{j,2p} = m_{j,2p} + p - j. \tag{20}$$

The operator $T_{\epsilon,\mathbf{m}_n}(I_{2p+1,2p})$ of the representation $T_{\epsilon,\mathbf{m}_n}$ of $U_q(\mathrm{so}_n)$ acts upon our basis elements, labelled by (17), by the formulas

$$T_{\epsilon,\mathbf{m}_n}(I_{2p+1,2p})|\xi_n\rangle = \delta_{m_{p,2p},1/2}\, \frac{\epsilon_{2p+1}}{q^{1/2} - q^{-1/2}} D_{2p}(\xi_n)|\xi_n\rangle +$$

$$+ \sum_{j=1}^{p} \frac{A^j_{2p}(\xi_n)}{q^{l_{j,2p}} - q^{-l_{j,2p}}} |(\xi_n)^{+j}_{2p}\rangle - \sum_{j=1}^{p} \frac{A^j_{2p}((\xi_n)^{-j}_{2p})}{q^{l_{j,2p}} - q^{-l_{j,2p}}} |(\xi_n)^{-j}_{2p}\rangle, \tag{21}$$

where the summation in the last sum must be from 1 to $p-1$ if $m_{p,2p} = 1/2$, and the operator $T_{\mathbf{m}_n}(I_{2p,2p-1})$ of the representation $T_{\mathbf{m}_n}$ acts as

$$T_{\epsilon,\mathbf{m}_n}(I_{2p,2p-1})|\xi_n\rangle = \sum_{j=1}^{p-1} \frac{B^j_{2p-1}(\xi_n)}{[2l_{j,2p-1} - 1][l_{j,2p-1}]_+} |(\xi_n)^{+j}_{2p-1}\rangle -$$

$$- \sum_{j=1}^{p-1} \frac{B^j_{2p-1}((\xi_n)^{-j}_{2p-1})}{[2l_{j,2p-1} - 1][l_{j,2p-1} - 1]_+} |(\xi_n)^{-j}_{2p-1}\rangle + \epsilon_{2p} \hat{C}_{2p-1}(\xi_n)|\xi_n\rangle, \tag{22}$$

where

$$[a]_+ = \frac{q^a + q^{-a}}{q - q^{-1}}.$$

In these formulas, $(\xi_n)^{\pm j}_k$ means the tableau (17) in which j-th component $m_{j,k}$ in $\mathbf{m}_k$ is replaced by $m_{j,k} \pm 1$. Matrix elements A^j_{2p} and B^j_{2p-1} are given by the same formulas as in (10) and (11) (that is, by the formulas (12) and (13)) and

$$\hat{C}_{2p-1}(\xi_n) = \frac{\prod_{s=1}^{p}[l_{s,2p}]_+ \prod_{s=1}^{p-1}[l_{s,2p-2}]_+}{\prod_{s=1}^{p-1}[l_{s,2p-1}]_+[l_{s,2p-1} - 1]_+}. \tag{23}$$

$$D_{2p}(\xi_n) = \frac{\prod_{i=1}^{p}[l_{i,2p+1} - \frac{1}{2}] \prod_{i=1}^{p-1}[l_{i,2p-1} - \frac{1}{2}]}{\prod_{i=1}^{p-1}[l_{i,2p} + \frac{1}{2}][l_{i,2p} - \frac{1}{2}]}. \tag{24}$$

For the operators $T_{\epsilon,\mathbf{m}_n}(I_{3,2})$ and $T_{\epsilon,\mathbf{m}_n}(I_{2,1})$ we have

$$T_{\epsilon,\mathbf{m}_n}(I_{3,2})|\xi_n\rangle = \frac{1}{q^{m_{1,2}}-q^{-m_{1,2}}}(([l_{1,3}+m_{1,2}][l_{1,3}-m_{1,2}-1])^{1/2}|(\xi_n)_2^{+1}\rangle -$$

$$([l_{1,3}+m_{1,2}-1][l_{1,3}-m_{1,2}])^{1/2}|(\xi_n)_2^{-1}\rangle)$$

if $m_{1,2}\ne \frac{1}{2}$,

$$T_{\epsilon,\mathbf{m}_n}(I_{3,2})|\xi_n\rangle = \frac{1}{q^{1/2}-q^{-1/2}}(\epsilon_3[l_{1,3}-1/2]|(\xi_n)\rangle + ([l_{1,3}+1/2][l_{1,3}-3/2])^{1/2}|(\xi_n)_2^{+1}\rangle)$$

if $m_{1,2}=\frac{1}{2}$, and

$$T_{\epsilon,\mathbf{m}_n}(I_{2,1})|\xi_n\rangle = \epsilon_2[m_{1,2}]_+|\xi_n\rangle .$$

The fact that the above operators $T_{\epsilon,\mathbf{m}_n}(I_{k,k-1})$ satisfy the defining relations (1)–(3) of the algebra $U'_q(\mathrm{so}_n)$ is proved in the following way. We take the formulas (10)–(14) for the classical type representations $T_{\mathbf{m}_n}$ of $U'_q(\mathrm{so}_n)$ with half-integral $m_{i,n}$ and replace there every $m_{j,2p+1}$ by $m_{j,2p+1}-\mathrm{i}\pi/2h$, every $m_{j,2p}$, $j\ne p$, by $m_{j,2p}-\mathrm{i}\pi/2h$ and $m_{p,2p}$ by $m_{p,2p}-\epsilon_2\epsilon_4\cdots\epsilon_{2p}\mathrm{i}\pi/2h$, where each ϵ_{2s} is equal to $+1$ or -1 and h is defined by $q=e^h$. Repeating almost word by word the reasoning of the paper [12], we prove that the operators given by formulas (10)–(14) satisfy the defining relations (1)–(3) of the algebra $U'_q(\mathrm{so}_n)$ after this replacement. Therefore, these operators determine a representation of $U'_q(\mathrm{so}_n)$. We denote this representation by $T'_{\mathbf{m}_n}$. After a simple rescaling, the operators $T'_{\mathbf{m}_n}(I_{k,k-1})$ take the form

$$T'_{\mathbf{m}_n}(I_{2p+1,2p})|\xi_n\rangle = \sum_{j=1}^{p}\frac{A^j_{2p}(\xi_n)}{q^{l_{j,2p}}-q^{-l_{j,2p}}}|(\xi_n)^{+j}_{2p}\rangle - \sum_{j=1}^{p}\frac{A^j_{2p}((\xi_n)^{-j}_{2p})}{q^{l_{j,2p}}-q^{-l_{j,2p}}}|(\xi_n)^{-j}_{2p}\rangle ,$$

$$T'_{\mathbf{m}_n}(I_{2p,2p-1})|\xi_n\rangle = \sum_{j=1}^{p-1}\frac{B^j_{2p-1}(\xi_n)}{[2l_{j,2p-1}-1][l_{j,2p-1}]_+}|(\xi_n)^{+j}_{2p-1}\rangle -$$

$$-\sum_{j=1}^{p-1}\frac{B^j_{2p-1}((\xi_n)^{-j}_{2p-1})}{[2l_{j,2p-1}-1][l_{j,2p-1}-1]_+}|(\xi_n)^{-j}_{2p-1}\rangle + \epsilon_{2p}\hat{C}_{2p-1}(\xi_n)|\xi_n\rangle ,$$

where A^j_{2p}, B^j_{2p-1} and $\hat{C}_{2p-1}$ are such as in the formulas (21) and (22). The representations $T'_{\mathbf{m}_n}$ are reducible. We decompose these representations into subrepresentations in the following way. We fix p ($p=1,2,\cdots,\{(n-1)/2\}$) and decompose the space $\mathcal{H}$ of the representation $T'_{\mathbf{m}_n}$ into direct sum of two subspaces $\mathcal{H}_{\epsilon_{2p+1}}$, $\epsilon_{2p+1}=\pm 1$, spanned by the basis vectors

$$|\xi_n\rangle_{\epsilon_{2p+1}} = |\xi_n\rangle - \epsilon_{2p+1}|\xi'_n\rangle, \qquad m_{p,2p}\ge 1/2,$$

respectively, where $|\xi'_n\rangle$ is obtained from $|\xi_n\rangle$ by replacement of $m_{p,2p}$ by $-m_{p,2p}$. A direct verification shows that two subspaces $\mathcal{H}_{\epsilon_{2p+1}}$ are invariant with respect to all the operators $T'_{\mathbf{m}_n}(I_{k,k-1})$. Now we take the subspaces $\mathcal{H}_{\epsilon_{2p+1}}$ and repeat the same procedure for some s, $s\ne p$, and decompose each of these subspaces into two invariant subspaces. Continuing this procedure further we decompose the representation space $\mathcal{H}$ into a direct sum of $2^{\{(n-1)/2\}}$ invariant subspaces. The operators $T'_{\mathbf{m}_n}(I_{k,k-1})$ act upon these subspaces by the formulas (21) and (22). We denote the corresponding subrepresentations on these subspaces by $T_{\epsilon,\mathbf{m}_n}$. The above reasoning shows that the operators $T_{\epsilon,\mathbf{m}_n}(I_{k,k-1})$ satisfy the defining relations (1)–(3) of the algebra $U'_q(\mathrm{so}_n)$.

Theorem 1. *The representations $T_{\epsilon,\mathbf{m}_n}$ are irreducible. The representations $T_{\epsilon,\mathbf{m}_n}$ and $T_{\epsilon',\mathbf{m}'_n}$ are pairwise nonequivalent for $(\epsilon,\mathbf{m}_n)\neq(\epsilon',\mathbf{m}'_n)$. For any admissable $(\epsilon,\mathbf{m}_n)$ and $\mathbf{m}'_n$ the representations $T_{\epsilon,\mathbf{m}_n}$ and $T_{\mathbf{m}'_n}$ are pairwise nonequivalent.*

The algebra $U'_q(\mathrm{so}_n)$ has non-trivial one-dimensional representations. They are special cases of the representations of the nonclassical type. They are described as follows.

Let $\epsilon:=(\epsilon_2,\epsilon_3,\cdots,\epsilon_n)$, $\epsilon_i=\pm1$, and let $\mathbf{m}_n=(m_{1,n},m_{2,n},\cdots,m_{\{n/2\},n})=(\frac12,\frac12,\cdots,\frac12)$. Then the corresponding representations $T_{\epsilon,\mathbf{m}_n}$ are one-dimensional and are given by the formulas

$$T_{\epsilon,\mathbf{m}_n}(I_{k+1,k})|\xi_n\rangle=\frac{\epsilon_{k+1}}{q^{1/2}-q^{-1/2}}.$$

Thus, to every $\epsilon:=(\epsilon_2,\epsilon_3,\cdots,\epsilon_n)$, $\epsilon_i=\pm1$, there corresponds a one-dimensional representation of $U'_q(\mathrm{so}_n)$.

5. Definition of representations of $U'_q(\mathrm{so}_{n,1})$ and $U_q(\mathrm{iso}_n)$

Let us recall that we assume that q is a positive number. We give the following definition of infinite dimensional representations of the algebras $U'_q(\mathrm{so}_{n,1})$ and $U_q(\mathrm{iso}_n)$ (we denote these algebras by $\mathcal{A}$). It is a homomorphism $R:\mathcal{A}\to\mathcal{L}(\mathcal{H})$ of $\mathcal{A}$ to the space $\mathcal{L}(\mathcal{H})$ of linear operators (bounded or unbounded) on a Hilbert space $\mathcal{H}$ such that

(a) operators $R(a)$, $a\in\mathcal{A}$, are defined on an invariant everywhere dense subspace $\mathcal{D}\subset\mathcal{H}$;

(b) $R\downarrow U'_q(\mathrm{so}_n)$ decomposes into a direct sum of irreducible finite dimensional representations of $U'_q(\mathrm{so}_n)$ (with finite multiplicities if R is irreducible);

(c) subspaces of irreducible representations of $U'_q(\mathrm{so}_n)$ belong to $\mathcal{D}$.

Two infinite dimensional irreducible representations R and R' of $\mathcal{A}$ on spaces $\mathcal{H}$ and $\mathcal{H}'$, respectively, are called (algebraically) equivalent if there exists an everywhere dense invariant subspaces $V\subset\mathcal{D}$ and $V'\subset\mathcal{D}'$ and a one-to-one linear operator $A:V\to V'$ such that $AR(a)v=R'(a)Av$ for all $a\in\mathcal{A}$ and $v\in V$.

Remark that our definition of infinite dimensional representations of $U'_q(\mathrm{so}_{n,1})$ and $U_q(\mathrm{iso}_n)$ corresponds to the definition of Harish-Chandra modules for the pairs $(\mathrm{so}_{n,1},\mathrm{so}_n)$ and $(\mathrm{iso}_n,\mathrm{so}_n)$, respectively. Thus, modules determined by representations of the above definition can be called q-Harish-Chandra modules of the pairs $(U'_q(\mathrm{so}_{n,1}),U'_q(\mathrm{so}_n))$ and $(U_q(\mathrm{iso}_n),U'_q(\mathrm{so}_n))$, respectively.

6. Classical type representations of $U_q(\mathrm{iso}_n)$

There are the following classes of irreducible representations of $U_q(\mathrm{iso}_n)$:

(a) Finite dimensional irreducible representations R of $U'_q(\mathrm{so}_n)$. They are irreducible representations of $U_q(\mathrm{iso}_n)$ with $R(T_n)=0$.

(b) Infinite dimensional irreducible representations of the classical type.

(c) Infinite dimensional irreducible representations of the nonclassical type.

Let us describe representations of class (b). They are given by non-zero complex parameter λ and by numbers $\mathbf{m}=(m_{2,n+1},m_{3,n+2},\cdots,m_{\{(n+1)/2\},n+1})$ describing irreducible representations of the classical type of the subalgebra $U'_q(\mathrm{so}_{n-1})$ (see [15] and [16]). We denote the corresponding representations of $U_q(\mathrm{iso}_n)$ by $R_{\lambda\mathbf{m}}$.

In order to describe the space of the representation $R_{\lambda\mathbf{m}}$ we note that

$$R_{\lambda\mathbf{m}}\downarrow U'_q(\mathrm{so}_n)=\bigoplus\nolimits_{\mathbf{m}_n}T_{\mathbf{m}_n},\quad \mathbf{m}_n=(m_{1,n},\cdots,m_{\{n/2\},n}), \tag{26}$$

where the summation is over all irreducible representations $T_{\mathbf{m}_n}$ of $U'_q(\mathrm{so}_n)$ which contain the irreducible representation of $U'_q(\mathrm{so}_{n-1})$ given by the numbers $\mathbf{m}$, that is, such that

$$m_{1,n} \ge m_{2,n+1} \ge m_{2,n} \ge m_{3,n+1} \ge m_{3,n} \ge \cdots .$$

The carrier space $\hat{\mathcal{H}}_{\mathbf{m}}$ of the representation $R_{\lambda\mathbf{m}}$ decomposes as $\hat{\mathcal{H}}_{\mathbf{m}} = \bigoplus_{\mathbf{m}_n} \mathcal{H}_{\mathbf{m}_n}$, where the summation is such as in (26) and $\mathcal{H}_{\mathbf{m}_n}$ are the subspaces, where the representations $T_{\mathbf{m}_n}$ of $U'_q(\mathrm{so}_n)$ are realized. We choose the Gel'fand–Tsetlin basis in every subspace $\mathcal{H}_{\mathbf{m}_n}$. The set of all these Gel'fand–Tsetlin bases gives a basis of the space $\hat{\mathcal{H}}_{\mathbf{m}}$. We denote the basis elements by $|\mathbf{m}_n, M\rangle$, where M are the corresponding Gel'fand–Tsetlin tableaux. The numbers m_{ij} from $|\mathbf{m}_n, M\rangle$ determine the numbers l_{ij} as in section 3. The numbers $m_{i,n+1}$ determine the numbers

$$l_{i,2k+1} = m_{i,2k+1} + k - i + 1, \quad n = 2k, \qquad l_{i,2k} = m_{i,2k} + k - i, \quad n = 2k-1.$$

The operators $R_{\lambda\mathbf{m}}(I_{i,i-1})$ are given by formulas of the classical type representations of the algebra $U'_q(\mathrm{so}_n)$ given in section 3. For the operators $R_{\lambda\mathbf{m}}(T_{2k})$ and $R_{\lambda\mathbf{m}}(T_{2k-1})$ we have the expressions

$$R_{\lambda\mathbf{m}}(T_{2k})|\mathbf{m}_{2k}, M\rangle = \lambda \sum_{j=1}^{k} \frac{\tilde{A}^j_{2k}(\mathbf{m}_{2k}, M)}{q^{l_{j,2k}} + q^{-l_{j,2k}}} |\mathbf{m}^{+j}_{2k}, M\rangle +$$

$$+\lambda \sum_{j=1}^{k} \frac{\tilde{A}^j_{2k}(\mathbf{m}^{-j}_{2k}, M)}{q^{l_{j,2k}} + q^{-l_{j,2k}}} |m^{-j}_{2k}, M\rangle, \tag{27}$$

$$R_{\lambda\mathbf{m}}(T_{2k-1})|\mathbf{m}_{2k-1}, M\rangle = \lambda \sum_{j=1}^{k-1} \frac{\tilde{B}^j_{2k-1}(\mathbf{m}_{2k-1}, M)}{[2l_{j,2k-1} - 1][l_{j,2k-1}]} |\mathbf{m}^{+j}_{2k-1}, M\rangle +$$

$$+\lambda \sum_{j=1}^{k-1} \frac{\tilde{B}^j_{2k-1}(\mathbf{m}^{-j}_{2k-1}, M)}{[2l_{j,2k-1} - 1][l_{j,2k-1} - 1]} |\mathbf{m}^{-j}_{2k-1}, M\rangle + \lambda \tilde{C}_{2k-1}(\mathbf{m}_{2k-1}, M)|\mathbf{m}_{2k-1}, M\rangle, \tag{28}$$

where $\mathbf{m}^{\pm j}_r$ means the set of numbers $\mathbf{m}_r$ with $m_{j,r}$ replaced by $m_{j,r} \pm 1$, respectively, the coefficients are given by

$$\tilde{A}^j_{2k}(\mathbf{m}_{2k}, M) = \left(\frac{\prod_{i=2}^{k} [l_{i,2k+1} + l_{j,2k}][l_{i,2k+1} - l_{j,2k} - 1]}{\prod_{i \ne j} [l_{i,2k} + l_{j,2k}][l_{i,2k} - l_{j,2k}]} \times \right.$$

$$\left. \times \frac{\prod_{i=1}^{k-1} [l_{i,2k-1} + l_{j,2k}][l_{i,2k-1} - l_{j,2k} - 1]}{\prod_{i \ne j} [l_{i,2k} + l_{j,2k} + 1][l_{i,2k} - l_{j,2k} - 1]} \right)^{1/2},$$

$$\tilde{B}^j_{2k-1}(\mathbf{m}_{2k-1}, M) = \left(\frac{\prod_{i=2}^{k} [l_{i,2k} + l_{j,2k-1}][l_{i,2k} - l_{j,2k-1}]}{\prod_{i \ne j} [l_{i,2k-1} + l_{j,2k-1}][l_{i,2k-1} - l_{j,2k-1}]} \times \right.$$

$$\left. \times \frac{\prod_{i=1}^{k-1} [l_{i,2k-2} + l_{j,2k-1}][l_{i,2k-2} - l_{j,2k-1}]}{\prod_{i \ne j} [l_{i,2k-1} + l_{j,2k-1} - 1][l_{i,2k-1} - l_{j,2k-1} - 1]} \right)^{1/2},$$

$$\tilde{C}_{2k-1}(M) = \frac{\prod_{s=2}^{k} [l_{s,2k}] \prod_{s=1}^{k-1} [l_{s,2k-2}]}{\prod_{s=1}^{k-1} [l_{s,2k-1}][l_{s,2k-1} - 1]}.$$

These formulas were given in [16] (see also [15]).

The representations $R_{\lambda\mathbf{m}}$, $\lambda\ne 0$, are irreducible. The representations $R_{\lambda\mathbf{m}}$ and $R_{\mu\mathbf{m}'}$ are equivalent if and only if $\mathbf{m}=\mathbf{m}'$ and $\lambda=\pm\mu$. The operator $R_{\lambda\mathbf{m}}(T_n)$ is bounded.

7. Nonclassical type representations of $U_q(\mathrm{iso}_n)$

Now we describe irreducible representations of nonclassical type (that is, representations R for which there exists no limit $q\to 1$ for the operators $R(T_n)$ and $R(I_{i,i-1})$). These representations are given by $\epsilon := (\epsilon_2,\epsilon_3,\cdots,\epsilon_{n+1})$, non-zero complex parameter λ and by numbers $\mathbf{m}=(m_{2,n+1},m_{3,n+2},\cdots,m_{\{(n+1)/2\},n+1})$, $m_{2,n+1}\ge m_{3,n+2}\ge\cdots\ge m_{\{(n+1)/2\},n+1}\ge 1/2$, describing irreducible representations of the nonclassical type of the subalgebra $U'_q(\mathrm{so}_{n-1})$ (see section 4). We denote the corresponding representations of $U_q(\mathrm{iso}_n)$ by $R_{\epsilon,\lambda,\mathbf{m}}$.

In order to describe the space of the representation $R_{\epsilon,\lambda,\mathbf{m}}$ we note that

$$R_{\epsilon,\lambda,\mathbf{m}}\downarrow U'_q(\mathrm{so}_n)=\bigoplus_{\mathbf{m}_n} T_{\epsilon',\mathbf{m}_n},\quad \mathbf{m}_n=(m_{1,n},\cdots,m_{\{n/2\},n}), \tag{29}$$

where $\epsilon'=(\epsilon_2,\cdots,\epsilon_n)$ is the part of the set ϵ, the summation is over all irreducible nonclassical type representations $T_{\epsilon',\mathbf{m}_n}$ of $U'_q(\mathrm{so}_n)$ for which the components of $\mathbf{m}_n$ satisfy the "betweenness" conditions

$$m_{1,2k+1}\ge m_{1,2k}\ge m_{2,2k+1}\ge m_{2,2k}\ge ...\ge m_{k,2k+1}\ge m_{k,2k}\ge 1/2 \ \text{ if }\ n=2k,$$

$$m_{1,2k}\ge m_{1,2k-1}\ge m_{2,2k}\ge m_{2,2k-1}\ge ...\ge m_{k-1,2k-1}\ge m_{k,2k}\ \text{ if }\ n=2k-1.$$

The carrier space $\hat{\mathcal{H}}_{\epsilon,\mathbf{m}}$ of the representation $R_{\epsilon,\lambda,\mathbf{m}}$ decomposes as $\hat{\mathcal{H}}_{\epsilon,\mathbf{m}}=\bigoplus_{\mathbf{m}_n}\mathcal{H}_{\epsilon',\mathbf{m}_n}$, where the summation is such as in (29) and $\mathcal{H}_{\epsilon',\mathbf{m}_n}$ are the subspaces, where the representations $T_{\epsilon',\mathbf{m}_n}$ of $U'_q(\mathrm{so}_n)$ are realized. We choose a basis in every subspace $\mathcal{H}_{\epsilon',\mathbf{m}_n}$ as in section 4. The set of all these bases gives a basis of the space $\hat{\mathcal{H}}_{\epsilon,\mathbf{m}}$. We denote the basis elements by $|\mathbf{m}_n,M\rangle$, where M are the corresponding tableaux. The numbers m_{ij} from $|\mathbf{m}_n,M\rangle$ determine the numbers l_{ij} as in section 3. The numbers $m_{i,n+1}$ determine the numbers $l_{i,n+1}$ as in section 6. The operators $R_{\epsilon,\lambda,\mathbf{m}}(I_{i,i-1})$ are given by formulas of the nonclassical type representations of the algebra $U'_q(\mathrm{so}_n)$ from section 4. For the operators $R_{\epsilon,\lambda,\mathbf{m}}(T_{2k})$ and $R_{\epsilon,\lambda,\mathbf{m}}(T_{2k-1})$ we have the expressions

$$R_{\epsilon,\lambda,\mathbf{m}}(T_{2k-1})|\mathbf{m}_{2k-1},M\rangle=\lambda\sum_{j=1}^{k-1}\frac{\tilde{B}^j_{2k-1}(\mathbf{m}_{2k-1},M)}{[2l_{j,2k-1}-1][l_{j,2k-1}]_+}|\mathbf{m}^{+j}_{2k-1},M\rangle+$$

$$+\lambda\sum_{j=1}^{k-1}\frac{\tilde{B}^j_{2k-1}(\mathbf{m}^{-j}_{2k-1},M)}{[2l_{j,2k-1}-1][l_{j,2k-1}-1]_+}|\mathbf{m}^{-j}_{2k-1},M\rangle+\mathrm{i}\epsilon_{2k}\lambda\hat{C}_{2k-1}(\mathbf{m}_{2k-1},M)|\mathbf{m}_{2k-1},M\rangle,$$

$$R_{\epsilon,\lambda,\mathbf{m}}(T_{2k})|\mathbf{m}_{2k},M\rangle=\mathrm{i}\lambda\delta_{m_{p,2p},1/2}\,\frac{\epsilon_{2k+1}}{q^{1/2}-q^{-1/2}}D_{2k}|\mathbf{m}_{2k},M\rangle+$$

$$+\lambda\sum_{j=1}^{k}\frac{\tilde{A}^j_{2k}(\mathbf{m}_{2k},M)}{q^{l_{j,2k}}-q^{-l_{j,2k}}}|\mathbf{m}^{+j}_{2k},M\rangle+\lambda\sum_{j=1}^{k}\frac{\tilde{A}^j_{2k}(\mathbf{m}^{-j}_{2k},M)}{q^{l_{j,2k}}-q^{-l_{j,2k}}}|\mathbf{m}^{-j}_{2k},M\rangle,$$

where the summation in the last sum must be from 1 to $k-1$ if $m_{k,2k}=1/2$, and $\tilde{A}^j_{2k}$, $\tilde{B}^j_{2k-1}$ are such as in (27) and (28), and

$$\hat{C}_{2k-1}(M)=\frac{\prod_{s=2}^{k}[l_{s,2k}]_+\prod_{s=1}^{k-1}[l_{s,2k-2}]_+}{\prod_{s=1}^{k-1}[l_{s,2k-1}]_+[l_{s,2k-1}-1]_+},$$

$$D_{2k}=\frac{\prod_{i=2}^{k}[l_{i,2k+1}-\frac12]\prod_{i=1}^{k-1}[l_{i,2k-1}-\frac12]}{\prod_{i=1}^{k-1}[l_{i,2k}+\frac12][l_{i,2k}-\frac12]}. \tag{30}$$

Theorem 2. *The representations $R_{\epsilon,\lambda,\mathbf{m}}$ are irreducible. The representations $R_{\epsilon,\lambda,\mathbf{m}}$ and $R_{\epsilon',\lambda',\mathbf{m}'}$ are equivalent if and only if $\epsilon=\epsilon'$, $\mathbf{m}=\mathbf{m}'$ and $\lambda=\pm\lambda'$. The operators $R_{\epsilon,\lambda,\mathbf{m}}(T_n)$ are bounded. The representation $R_{\epsilon,\lambda,\mathbf{m}}$ is equivalent to no of the representations $R_{\lambda',\mathbf{m}'}$ of section 6.*

8. Classical type representations of $U'_q(\mathrm{so}_{n,1})$

Let us describe the principal series representations of the algebra $U'_q(\mathrm{so}_{n,1})$. They are given by a complex parameter c and by numbers $\mathbf{m}=(m_{2,n+1},m_{3,n+1},\cdots,m_{\{(n+1)/2\},n+1})$ describing irreducible representations of the classical type of the subalgebra $U'_q(\mathrm{so}_{n-1})$ (see [8] and [20]). We denote the corresponding representations of $U_q(\mathrm{so}_{n,1})$ by $R_{c,\mathbf{m}}$.

In order to describe the space of the representation $R_{c,\mathbf{m}}$ we note that

$$R_{c,\mathbf{m}}\downarrow U'_q(\mathrm{so}_n)=\bigoplus\nolimits_{\mathbf{m}_n}T_{\mathbf{m}_n},\quad \mathbf{m}_n=(m_{1,n},\cdots,m_{\{n/2\},n}), \tag{31}$$

where the summation is over all irreducible representations $T_{\mathbf{m}_n}$ of $U'_q(\mathrm{so}_n)$ which contain the irreducible representation of $U'_q(\mathrm{so}_{n-1})$ given by the numbers $\mathbf{m}$, that is,

$$m_{1,n}\ge m_{2,n+1}\ge m_{2,n}\ge m_{3,n+1}\ge\cdots.$$

Thus, the carrier space $\hat{\mathcal{H}}_{\mathbf{m}}$ of the representation $R_{c,\mathbf{m}}$ decomposes as $\hat{\mathcal{H}}_{\mathbf{m}}=\bigoplus_{\mathbf{m}_n}\mathcal{H}_{\mathbf{m}_n}$, where the summation is such as in (31) and $\mathcal{H}_{\mathbf{m}_n}$ are the subspaces, where the representations $T_{\mathbf{m}_n}$ of $U'_q(\mathrm{so}_n)$ are realized. We choose the Gel'fand–Tsetlin basis in every subspace $\mathcal{H}_{\mathbf{m}_n}$. The set of all these Gel'fand–Tsetlin bases gives a basis of the space $\hat{\mathcal{H}}_{\mathbf{m}}$. We denote the basis elements by $|\mathbf{m}_n,M\rangle$, where M are the corresponding Gel'fand–Tsetlin tableaux. The numbers m_{ij} from $|\mathbf{m}_n,M\rangle$ determine the numbers l_{ij} as in section 3. The numbers $m_{i,n+1}$ determine the numbers

$$l_{i,2k+1}=m_{i,2k+1}+k-i+1,\quad n=2k,\qquad l_{i,2k}=m_{i,2k}+k-i,\quad n=2k-1.$$

The operators $R_{c,\mathbf{m}}(I_{i+1,i})$, $i<n$, are given by formulas of representations of the algebra $U'_q(\mathrm{so}_n)$ given in section 3. For the action of the operators $R_{c,\mathbf{m}}(I_{2k+1,2k})$ and $R_{c,\mathbf{m}}(I_{2k,2k-1})$ we have the expressions

$$R_{c,\mathbf{m}}(I_{2k+1,2k})|\mathbf{m}_{2k},M\rangle=\sum_{j=1}^{k}([c+l_{j,2k}][c-l_{j,2k}-1])^{1/2}\frac{\tilde{A}^j_{2k}(\mathbf{m}_{2k},M)}{q^{l_{j,2k}}+q^{-l_{j,2k}}}|\mathbf{m}^{+j}_{2k},M_{2k}\rangle-$$

$$-\sum_{j=1}^{k}([c+l_{j,2k}-1][c-l_{j,2k}])^{1/2}\frac{\tilde{A}^j_{2k}(\mathbf{m}^{-j}_{2k},M)}{q^{l_{j,2k}}+q^{-l_{j,2k}}}|m^{-j}_{2k},M\rangle, \tag{32}$$

$$R_{c,\mathbf{m}}(I_{2k,2k-1})|\mathbf{m}_{2k-1},M\rangle=\sum_{j=1}^{k-1}([c+l_{j,2k-1}][c-l_{j,2k-1}])^{1/2}\frac{\tilde{B}^j_{2k-1}(\mathbf{m}_{2k-1},M)}{[2l_{j,2k-1}-1][l_{j,2k-1}]}|\mathbf{m}^{+j}_{2k-1},M\rangle-$$

$$-\sum_{j=1}^{k-1}([c+l_{j,2k-1}-1][c-l_{j,2k-1}+1])^{1/2}\frac{\tilde{B}^j_{2k-1}(\mathbf{m}^{-j}_{2k-1},M)}{[2l_{j,2k-1}-1][l_{j,2k-1}-1]}|\mathbf{m}^{-j}_{2k-1},M\rangle+$$

$$+\mathrm{i}[c]\tilde{C}_{2k-1}(\mathbf{m}_{2k-1},M)|\mathbf{m}_{2k-1},M\rangle, \tag{33}$$

where $\tilde{A}^j_{2k}(\mathbf{m}_{2k}, M)$, $\tilde{B}^j_{2k-1}(\mathbf{m}_{2k-1}, M)$ and $\tilde{C}_{2k-1}(\mathbf{m}_{2k-1}, M)$ are such as in (27) and (28). These formulas were given in [20] (see also [8]).

Theorem 2. *The representation $R_{c,\mathbf{m}}$ of $U'_q(\mathrm{so}_{2k,1})$ is irreducible if and only if c is not integer (resp. half-integer) if $l_{j,2k+1}$, $j = 2, 3, \cdots, k$, are integers (resp. half-integers) or one of the numbers c, $1 - c$ coincides with one of the numbers $l_{j,2k+1}$, $j = 2, 3, \cdots, k$. The representation $R_{c,\mathbf{m}}$ of $U'_q(\mathrm{so}_{2k-1,1})$ is irreducible if and only if c is not integer (resp. half-integer) if $l_{j,2k}$, $j = 2, 3, \cdots, k$, are integers (resp. half-integers) or $|c|$ coincides with one of the numbers $l_{j,2k}$, $j = 2, 3, \cdots, k$, or $|c| < |l_{k,2k}|$.*

The reducible representations $R_{c,\mathbf{m}}$ can be analysed as in the case of the principal nonunitary series of the algebra $\mathrm{so}_{n,1}$ (see, for example, [19]). It will be made in a separate paper.

9. Nonclassical type representations of $U'_q(\mathrm{so}_{n,1})$

Now we describe irreducible representations of nonclassical type (that is, representations R for which there exists no limit $q \to 1$ for the operators $R(I_{i,i-1})$). These representations are given by $\epsilon := (\epsilon_2, \epsilon_3, \cdots, \epsilon_{n+1})$, by a complex parameter c and by numbers $\mathbf{m} = (m_{2,n+1}, m_{3,n+1}, \cdots, m_{\{(n+1)/2\},n+1})$, $m_{2,n+1} \geq m_{3,n+2} \geq \cdots \geq m_{\{(n+1)/2\},n+1} \geq 1/2$, describing irreducible representations of the nonclassical type of the subalgebra $U'_q(\mathrm{so}_{n-1})$ (see section 4). We denote the corresponding representations of $U_q(\mathrm{so}_{n,1})$ by $R_{\epsilon,c,\mathbf{m}}$.

In order to describe the space of the representation $R_{\epsilon,c,\mathbf{m}}$ we note that

$$R_{\epsilon,\lambda,\mathbf{m}} \downarrow U'_q(\mathrm{so}_n) = \bigoplus_{\mathbf{m}_n} T_{\epsilon',\mathbf{m}_n}, \quad \mathbf{m}_n = (m_{1,n}, \cdots, m_{\{n/2\},n}), \tag{34}$$

where $\epsilon' = (\epsilon_2, \cdots, \epsilon_n)$, the summation is over all irreducible nonclassical type representations $T_{\epsilon',\mathbf{m}_n}$ of the subalgebra $U'_q(\mathrm{so}_n)$ for which the components of $\mathbf{m}_n$ satisfy the "betweenness" conditions

$$m_{1,2k+1} \geq m_{1,2k} \geq m_{2,2k+1} \geq m_{2,2k} \geq ... \geq m_{k,2k+1} \geq m_{k,2k} \geq 1/2 \ \text{ if } \ n = 2k \tag{35}$$

$$m_{1,2k} \geq m_{1,2k-1} \geq m_{2,2k} \geq m_{2,2k-1} \geq ... \geq m_{k-1,2k-1} \geq m_{k,2k} \ \text{ if } \ n = 2k-1. \tag{36}$$

The carrier space $\hat{\mathcal{H}}_{\epsilon,\mathbf{m}}$ of the representation $R_{\epsilon,c,\mathbf{m}}$ decomposes as $\hat{\mathcal{H}}_{\epsilon,\mathbf{m}} = \bigoplus_{\mathbf{m}_n} \mathcal{H}_{\epsilon,\mathbf{m}_n}$, where the summation is such as in (34) and $\mathcal{H}_{\epsilon',\mathbf{m}_n}$ are the subspaces, where the representations $T_{\epsilon',\mathbf{m}_n}$ of $U'_q(\mathrm{so}_n)$ are realized. We choose the basis in every subspace $\mathcal{H}_{\epsilon,\mathbf{m}_n}$ as in section 4. The set of all these bases gives a basis of the space $\hat{\mathcal{H}}_{\epsilon,\mathbf{m}}$. We denote the basis elements by $|\mathbf{m}_n, M\rangle$, where M are the corresponding tableaux. The numbers m_{ij} from $|\mathbf{m}_n, M\rangle$ determine the numbers l_{ij} as in section 3. The numbers $m_{i,n+1}$ determine the numbers $l_{i,n+1}$ as in section 8. The operators $R_{\epsilon,c,\mathbf{m}}(I_{i,i-1})$, $i \leq n$, are given by formulas of the nonclassical type representations of the algebra $U'_q(\mathrm{so}_n)$ given in section 4. For the operators $R_{\epsilon,c,\mathbf{m}}(I_{2k+1,2k})$ if $n = 2k$ and $R_{\epsilon,c,\mathbf{m}}(T_{2k,2k-1})$ if $n = 2k-1$ we have the expressions

$$R_{\epsilon,c,\mathbf{m}}(I_{2k,2k-1})|\mathbf{m}_{2k-1}, M\rangle =$$

$$= \sum_{j=1}^{k-1} ([c + l_{j,2k-1}][c - l_{j,2k}])^{1/2} \frac{\tilde{B}^j_{2k-1}(\mathbf{m}_{2k-1}, M)}{[2l_{j,2k-1} - 1][l_{j,2k-1}]_+} |\mathbf{m}^{+j}_{2k-1}, M\rangle -$$

$$- \sum_{j=1}^{k-1} ([c + l_{j,2k-1} + 1][c - l_{j,2k} + 1])^{1/2} \frac{\tilde{B}^j_{2k-1}(\mathbf{m}^{-j}_{2k-1}, M)}{[2l_{j,2k-1} - 1][l_{j,2k-1} - 1]_+} |\mathbf{m}^{-j}_{2k-1}, M\rangle +$$

$$+\epsilon_{2k}[c]_+\hat{C}_{2k-1}(\mathbf{m}_{2k-1},M)|\mathbf{m}_{2k-1},M\rangle,$$

$$R_{\epsilon,c,\mathbf{m}}(I_{2k+1,2k})|\mathbf{m}_{2k},M\rangle = \delta_{m_{k,2k},1/2}\,[c-1/2]\,\frac{\epsilon_{2k+1}}{q^{1/2}-q^{-1/2}}D_{2k}(m_{2k},M)|\mathbf{m}_{2k},M\rangle +$$

$$+\sum_{j=1}^{k}([c+l_{j,2k}][c-l_{j,2k}-1])^{1/2}\frac{\tilde{A}^j_{2k}(\mathbf{m}_{2k},M)}{q^{l_{j,2k}}-q^{-l_{j,2k}}}|\mathbf{m}^{+j}_{2k},M\rangle -$$

$$-\sum_{j=1}^{k}([c+l_{j,2k}-1][c-l_{j,2k}])^{1/2}\frac{\tilde{A}^j_{2k}(\mathbf{m}^{-j}_{2k},M)}{q^{l_{j,2k}}-q^{-l_{j,2k}}}|\mathbf{m}^{-j}_{2k},M\rangle,$$

where the summation in the last sum must be from 1 to $k-1$ if $m_{k,2k}=1/2$, and $\tilde{A}^j_{2k}$, $\tilde{B}^j_{2k-1}$ are such as in (27) and (28), and

$$\hat{C}_{2k-1}(m_{2k-1},M) = \frac{\prod_{s=2}^{k}[l_{s,2k}]_+\prod_{s=1}^{k-1}[l_{s,2k-2}]_+}{\prod_{s=1}^{k-1}[l_{s,2k-1}]_+[l_{s,2k-1}-1]_+},$$

$$D_{2k}(m_{2k},M) = \frac{\prod_{i=2}^{k}[l_{i,2k+1}-\frac{1}{2}]\prod_{i=1}^{k-1}[l_{i,2k-1}-\frac{1}{2}]}{\prod_{i=1}^{k-1}[l_{i,2k}+\frac{1}{2}][l_{i,2k}-\frac{1}{2}]}.$$

Theorem 3. *The representation $R_{\epsilon,c,\mathbf{m}}$ of $U'_q(\mathrm{so}_{2k,1})$ is irreducible if and only if c is not half-integer or one of the numbers c, $1-c$ coincides with one of the numbers $l_{j,2k+1}$, $j=2,3,\cdots,k$. The representation $R_{\epsilon,c,\mathbf{m}}$ of $U'_q(\mathrm{so}_{2k-1,1})$ is irreducible if and only if c is not half-integer or $|c|$ coincides with one of the numbers $l_{j,2k}$, $j=2,3,\cdots,k$, or $|c|<l_{k,2k}$.*

This theorem will be proved in a separate paper. There will be also studied reducible representations $R_{\epsilon,c,\mathbf{m}}$.

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